Cosmological Casimir effect with maximum planckian momentum and accelerating universe.

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Abstract

We develop here a mechanism that, without making use of a cosmological constant, reproduces an accelerating universe. This is done by taking into account Casimir vacuum energy density, assuming that the underlying theory allows a maximum momentum, that turns out to be the leading contribution term to Einstein equations in a large expanding FRW universe. As stated in numerous quantum gravity studies, we postulate that maximum momentum is related to the existence of the Planck length as a fundamental length. This insight, together with the assumption of a Planck scale correction to the energy/momentum dispersion-relation on a FRW background, is used here to calculate Casimir vacuum energy. We show that, under these hypothesis, an accelerated universe expansion is obtained. As last step we analyze the compatibility of the resulting model with experimental data, writing down the equation of state for Casimir energy and pressure and observing that this equation of state belongs to a class of models that naturally fits cosmological observations. We emphasize that our result relies, once a fundamental length is introduced in Casimir effect, just on general arguments thus it is independent on an explicit form of the energy-momentum dispersion relation.

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I.   INTRODUCTION.

As evident by consolidated astrophysical data, the universe is in accelerating expansion. To account for this fact, in the Λ-CDM model a cosmological constant has been introduced, with the meaning of Dark Energy (for a review see [1, 2, 3] and related refs.), that counts for about 70% of the total mass energy density. With the purpose of explaining the introduction of such a cosmological constant some authors have dealt with Casimir effect, but without making use of a maximum planckian momentum, see for exemple [4]. An alternative to the standard cosmological model is represented, for exemple, by modified gravity, as discussed in [5, 6, 7], and by phantom models, [8, 9] and related refs.

Here we want to present a mechanism for the accelerating universe that relies upon two hypothesis: 1) that a maximum momentum exists and that it is related to a fundamental scale, the Planck length; 2) that a Casimir vacuum energy contributes to the energy density of the universe.

There has been a long debate in quantum gravity literature (for a satisfactory review see [12]) about possible ways of introducing Planck length as a quantum gravity scale [10, 11].

Despite of severe difficulties in obtaining experimental data in which the role of Planck scale manifestly appears, in recent years a wide and increasing literature on quantum gravity phenomenology has been produced. Possible connections to forthcoming experiments have been explained and suitable observable quantities sensible to planckian structure of space-time have been identified [13, 14]. As a consequence, Planck scale corrections to the energy momentum dispersion relation have been considered as the straightest way to approach these phenomenological issues. Planck scale can in fact appear by heuristic considerations, either as a deformation parameter of the special relativistic or general relativistic energy momentum dispersion relations, or as a maximum spatial momentum [16, 17]. In this scenario, we first assume that analytical Planck scale deformation to the one particle energy momentum dispersion relation can be considered in a FRW expanding universe background. [1]

Some authors refer to the Planck length as a minimum observable length [15], that is responsible for cutting off trans-planckian degrees of freedom. Specially in Loop Quantum Gravity, recent works [18, 19] clarify how the very planckian structure of the space-time, emerging in a consistent quantum gravity model, may involve a qualitatively different evolution of the universe in the

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[1] To be simple, we will restrict our attention only to massless fields without any other internal degrees of freedom. This assumption will not affect in a drastic way our results, giving only small negligible corrections to the calculus shown below or supplying multiplication for order one constant extra factors.

(planckian) early times.

Anyway, even without making any strong assumptions on the deep planckian regime of a full quantum gravity theory, one can assume that if such a quantum gravity theory exists, Planck departures from standard classical gravitational relations can be also postulated at low energies or large distances [16, 21, 22]. Such a reasoning applies to the details of the analysis we are going to develop here below. In this letter we want to show that the introduction of the Planck length (although presented in a heuristic/phenomenological way, assuming Planck scale corrections to the one-particle energy momentum dispersion relation and hence to the Casimir vacuum energy density) may also induce considerable modification to the behavior of a large expanding universe, i.e. a universe whose metric conformal factor a satisfies $\lambda_{Planck} << a$. In section II, we in fact assume that a cosmological Casimir energy, calculated as a massless scalar field vacuum energy, contributes[2] to the right hand side of the Einstein equation. Vacuum energy is then calculated assuming Planck scale modified energy momentum dispersion relation for a single massless particle with a maximum planckian momentum. An analytical expansion in λ_{Planck}/a to the Casimir energy density is then performed. The reasonableness condition taken into account, $\lambda_{Planck} << a$, of course limits the validity of our reasonings to post-planckian cosmological eras.

By these hypothesis, in section III we obtain a mechanism that is different from the mechanisms of the models cited above, that can account for an accelerated expansion of the universe without postulating the existence of a cosmological constant. As last step, in section IV, we write down the equation of state for the Casimir energy density and pressure and the explicit form of the Casimir energy density. We then show that the model obtained belongs to a class of models that naturally fits cosmological observations. In section V, as closing remarks, we summarize the obtained results.

II. COSMOLOGICAL CASIMIR EFFECT WITH A MAXIMUM PLANCKIAN MOMENTUM.

To start with, let us consider an homogeneous space-time in a comooving FRW coordinates system. We can assume that the metric tensor is given by

$$ds^2 = -(cdt)^2 + a(t)^2[d\chi^2 + \Sigma^2(d\theta^2 + Sin^2(\theta)d\phi^2] \tag{1}$$

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[2] Such an assumption could be discussed in depth highlighting possible connections between classical and quantum field theories [20].

In this coordinates system, Einstein equations read

$$
\begin{cases}
G_{tt} = 3\left(\frac{\dot{a}}{ca}\right)^2 + 3\frac{k}{a^2} = \frac{8\pi G}{c^3}T_{tt} \\
\\
G_{\mu\mu} = -\frac{2\ddot{a}}{c^2 a} - \left(\frac{\dot{a}}{ca}\right)^2 - \frac{k}{a^2} = \frac{8\pi G}{c^3}T_{\mu\mu}
\end{cases}
\tag{2}
$$

where $\mu = \chi, \theta, \phi$ is the spatial index; $T_{\mu\mu}$ is the pressure of the system, T_{tt} is its energy density and $\frac{k}{a^2}$ is the spatial scalar curvature. As said in the introduction, maximum planckian momentum is introduced in the energy momentum dispersion relation, so this relation results modified. Using this hypothesis we can write the energy momentum relation for a massless scalar field in the form

$$
E = \hbar\,\omega(|\,\vec{k}\,|, \lambda, a)
\tag{3}
$$

We are considering for simplicity the case of a massless scalar field, but the following argument can be easily generalized to other kind of fields. The Casimir energy is given by

$$
T_{tt} = \frac{E_0}{cV} = \hbar \int \frac{d^3k}{c(2\pi)^3}\omega(k, \lambda, a) = \frac{2\hbar}{c(2\pi)^2}\int_{\frac{1}{2a},0}^{\frac{1}{\lambda}} dk k^2 \omega(k, \lambda, a) = \frac{\hbar}{2\pi^2}F(\lambda, a)
\tag{4}
$$

where

$$
F(\lambda, a) = \frac{1}{c}\int_{\frac{1}{2a},0}^{\frac{1}{\lambda}} dk k^2 \omega(k, \lambda, a)
$$

In this equation, the integration starts from 0 for the open or spatially flat FRW universe, from $\frac{1}{2a}$ for the closed FRW universe. Note that we are completely disregarding the contribution of matter and radiation that, as it will be evident later, is negligible in a large expanding universe. We then obtain for the system of Einstein equations

$$
\begin{cases}
\left(\frac{\dot{a}}{ca}\right)^2 + 3\frac{k}{a^2} = \frac{4\hbar}{3\pi c^3}\,G\,F(\lambda, a) \\
\\
(T_{tt}a^3)_{,t} = -T_{\mu\mu}(a^3)_{,t}
\end{cases}
\tag{5}
$$

Here we substituted the second equation of (2) with the energy conservation equation. The second equation in (5) is used just to determine $T_{\mu\mu}$, so at the moment we can ignore it. Let us now consider the function $F(\lambda, a)$. By dimensional analysis it follows that

$$
F(\lambda, a) = \frac{1}{\lambda^3}\left(\frac{\alpha(\frac{\lambda}{a})}{\lambda} + \frac{\beta(\frac{\lambda}{a})}{a}\right)
\tag{6}
$$

In fact we can write

$$
F(\lambda, a) = \frac{1}{c\lambda^3}\int_{\frac{\lambda}{2a},0}^{1} dx\, x^2\,\omega(\frac{x}{\lambda}, \lambda, a)
$$

Note that $\frac{\omega}{c}$ has the dimension of an inverse of length so that the only way to write it is

$$\frac{\omega}{c} = \sum_{k=0}^{\infty} \left(\frac{A_k(x)}{\lambda} + \frac{B_k(x)}{a} \right) \left(\frac{\lambda}{a} \right)^k$$

thus we have

$$F(\lambda, a) = \frac{1}{c\lambda^3} \int_{\frac{\lambda}{2a},\, 0}^{1} dx\, x^2\, \omega(\frac{x}{\lambda}, \lambda, a) = \frac{1}{\lambda^3} \sum_{k=0}^{\infty} \left(\frac{A_k}{\lambda} + \frac{B_k}{a} \right) \left(\frac{\lambda}{a} \right)^k = \frac{1}{\lambda^3} \left(\frac{\alpha(\frac{\lambda}{a})}{\lambda} + \frac{\beta(\frac{\lambda}{a})}{a} \right)$$

Here, α and β are analytic functions in $\frac{\lambda}{a}$.

$$\begin{cases} \alpha(\frac{\lambda}{a}) = \sum_{k=0}^{\infty} \alpha_k (\frac{\lambda}{a})^k \\ \beta(\frac{\lambda}{a}) = \sum_{k=0}^{\infty} \beta_k (\frac{\lambda}{a})^k \end{cases}$$

In order to obtain the net Casimir energy $F_{net}(\lambda, a)$, we have to subtract to this quantity its infinite limit

$$F(\lambda, \infty) = lim_{a \to \infty} F(\lambda, a) = \frac{\alpha(0)}{\lambda^4} \tag{7}$$

so that

$$F_{net}(\lambda, a) = \frac{1}{\lambda^3} \left(\frac{\alpha(\frac{\lambda}{a}) - \alpha(0)}{\lambda} + \frac{\beta(\frac{\lambda}{a})}{a} \right) = \frac{1}{\lambda^3 a} B(\frac{\lambda}{a}) \tag{8}$$

in which B is an analytic function of λ/a with $B(0) \neq 0$. This is our final expression for $F(\lambda, a)$.

III.   THE ACCELERATING UNIVERSE.

Now we can go back to the first Einstein equation and write it in the form:

$$\left(\frac{\dot{a}}{c} \right)^2 = -k + \frac{4\hbar G}{3\pi c^3} \frac{a}{\lambda^3} B(\frac{\lambda}{a}) \tag{9}$$

It is evident that this is an equation for an accelerating universe. Now we can set set $\lambda = \lambda_{Planck}$ and obtain

$$\left(\frac{\dot{a}}{c} \right)^2 = -k + \frac{4}{3\pi} \frac{a}{\lambda_{Planck}} B(\frac{\lambda_{Planck}}{a}) \tag{10}$$

For the leading term we find

$$\left(\frac{\dot{a}}{c} \right)^2 = -k + \frac{4}{3\pi} \frac{a}{\lambda_{Planck}} B(0) \tag{11}$$

Note that matter and radiation densities are completely negligible in a large universe, because of the fact that they are respectively of order $\frac{1}{a^3}$ and $\frac{1}{a^4}$. We want to stress that in order to write

this relation we used (3). Although we do not know explicitly (3) and hence the corresponding[3] $F(\lambda, a)$, we are able to predict the universe accelerating expansion in the limit of large $a(t)$. We also stress that this discussion is based on dimensional analysis, so it is, after the introduction of the fundamental Planck length, totally general. Now we can ask if relation (11) agree with cosmological data. To answer to this question we have to write the equation of state for Casimir Energy and Pressure and write the Casimir energy density as a function of $a(t)$.

IV. EQUATION OF STATE AND COSMOLOGICAL OBSERVATIONS.

To obtain equation of state for the Casimir Energy we have to use the second of equations in (5). Using (4) and (8) we have

$$T_{tt} = \frac{\hbar}{2\pi^2} \frac{B(\frac{\lambda}{a})}{a\lambda^3} \simeq \frac{\hbar}{2\pi^2} \frac{B(0)}{a\lambda^3} \tag{12}$$

In this approximation we have

$$T_{\mu\mu} = -\frac{2}{3} T_{tt} \tag{13}$$

We note that this result is in agreement with experimental data. In fact, as discussed in [3], equation of state with $-1 < \omega < 0$, where ω is the ratio of the pressure to the energy density, fits current cosmological observation best. So, as follows from the last equation, in our case we have $\omega = -\frac{2}{3}$, and this value belongs to the range mentioned above. Moreover, in order to confront our model with experimental data, we can link our parameter $B(0)$ with a_0 and H_0, respectively the scale factor today and the Hubble constant today. From (9), setting the spatial curvature equal to zero, in agreement with WMAP observations [23] [24], we have

$$
\begin{cases}
\rho_{casimir} = B(0)\rho_c \frac{a_0}{a} \\[2em]
a_0 = \frac{4c^2}{3\pi\lambda_{Planck} H_o^2} \\[2em]
\rho_c = \frac{3\hbar H_0^2}{8\pi c\lambda_{Planck}^2}
\end{cases}
\tag{14}
$$

where ρ_c is the critical energy density. It is evident that $B(0)$ simply represent the ratio between the Casimir energy density and the critical density. Note that $B(0)$ is a pure number thus it would

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[3] As an example, one may consider, without any particular physical intent, the case of the following energy/momentum dispersion relation $\omega(k, \lambda, a) = \frac{c}{\lambda} \ln\left(\frac{1}{1-\lambda k}\right)\left(1 + \frac{\lambda}{a}\right)$, from wich follows that $B(0) = \frac{11}{18}$

be desirable for it , following a naturalness criterion, to take values in the neighborhood of the unity. This is also in agreement with cosmological observations, that predict a value for this parameter close to $0.6 - 0.7$ [3]. In light of these facts, we conclude that our model is a good candidate to explain the accelerating expansion of the universe.

V.   CONCLUSIONS.

We conclude this letter remarking the fundamental points of our analysis. We first used the hypothesis of the existence of a maximum momentum related to the Planck scale and we calculated the Casimir energy density of a FRW expanding universe. This mechanism actually reproduces an accelerating universe. We want to emphasize that this result follows from dimensional analysis. At the end, we obtained the equation of state for Casimir energy and pressure and the expression of Casimir energy density as a function of the scale factor. These expressions are in agreement with current cosmological data. A further analysis is needed to study the compatibility of this toy model with CMB observations. In conclusion, this toy model can offer a mechanism to explain the accelerating expansion of the universe and it can be easily improved to give a real physical model, without affecting the fundamental result, by the inclusion of dark matter and other contributions to the total energy density.

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